Since our first goal is to describe the electric field produced by an atom or molecule, it will help to make some general observations about the electrostatic field external to any small system of charges.

10.2 The moments of a charge distribution

An atom or molecule consists of some electric charges occupying a small volume, perhaps a few cubic angstroms ($10^{-30}\,\mathrm{m}^3$) of space. We are interested in the electric field outside that volume, which arises from this rather complicated charge distribution. We shall be particularly concerned with the field far away from the source, by which we mean far away compared with the size of the source itself. What features of the charge structure mainly determine the field at remote points? To answer this, let's look at some arbitrary distribution of charges and see how we might go about computing the field at a point outside it. The discussion in this and the following section generalizes our earlier discussion of dipoles in Section 2.7.

Figure 10.3 shows a charge distribution of some sort located in the neighborhood of the origin of coordinates. It might be a molecule consisting of several positive nuclei and quite a large number of electrons. In any case we shall suppose it is described by a given charge density function $\rho(x, y, z)$; ρ is negative where the electrons are and positive where the nuclei are. To find the electric field at distant points we can begin by computing the potential of the charge distribution. To illustrate, let's take some point A out on the z axis. (Since we are not assuming any special symmetry in the charge distribution, there is nothing special about the z axis.) Let r be the distance of A from the origin. The electric potential at A, denoted by ϕ_A, is obtained as usual by adding the contributions from all elements of the charge distribution:

$$\phi_A = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(x', y', z')\,dv'}{R}. \tag{10.4}$$

In the integrand, dv' is an element of volume within the charge distribution, $\rho(x', y', z')$ is the charge density there, and R in the denominator is the distance from A to this particular charge element. The integration is carried out in the coordinates x', y', z', of course, and is extended over all the region containing charge. We can express R in terms of r and the distance r' from the origin to the charge element. Using the law of cosines with θ the angle between $\mathbf{r}'$ and the axis on which A lies, we have

$$R = (r^2 + r'^2 - 2rr'\cos\theta)^{1/2}. \tag{10.5}$$

With this substitution for R, the integral becomes

$$\phi_A = \frac{1}{4\pi\epsilon_0}\int \rho\,dv'(r^2 + r'^2 - 2rr'\cos\theta)^{-1/2}. \tag{10.6}$$

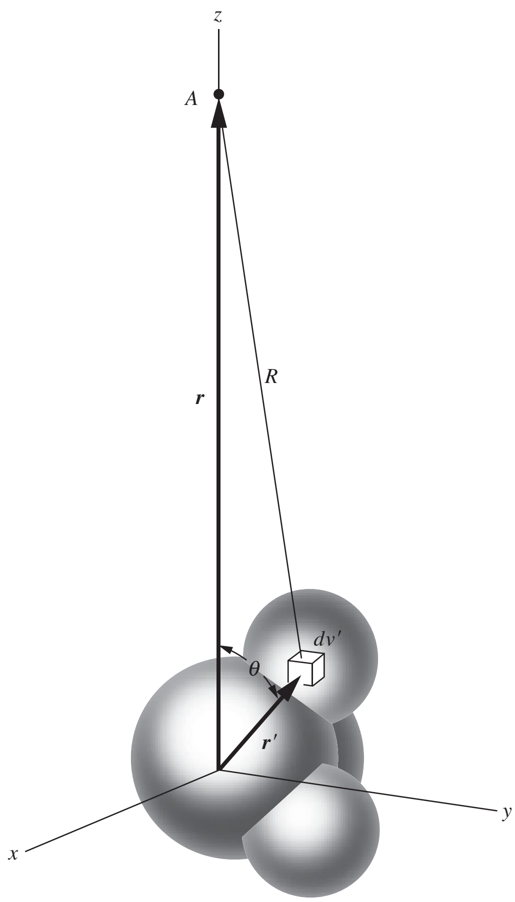


Figure 10.3.
Calculation of the potential, at a point A, of a molecular charge distribution.

Now we want to take advantage of the fact that, for a distant point like A, r' is much smaller than r for all parts of the charge distribution. This suggests that we should expand the square root in Eq. (10.5) in powers of r'/r. Writing

$$(r^2 + r'^2 - 2rr'\cos\theta)^{-1/2} = \frac{1}{r}\left[1 + \left(\frac{r'^2}{r^2} - \frac{2r'}{r}\cos\theta\right)\right]^{-1/2} \tag{10.7}$$

and using the expansion $(1+\delta)^{-1/2} = 1 - \delta/2 + 3\delta^2/8 - \cdots$, we get, after collecting together terms of the same power in r'/r, the following:

$$\begin{aligned}(r^2 + r'^2 - 2rr'\cos\theta)^{-1/2} \\ = \frac{1}{r}\left[1 + \frac{r'}{r}\cos\theta + \left(\frac{r'}{r}\right)^2 \frac{(3\cos^2\theta - 1)}{2} + \mathcal{O}\left[\left(\frac{r'}{r}\right)^3\right]\right],\end{aligned} \tag{10.8}$$

where the last term here indicates terms of order at least $(r'/r)^3$. These are very small if $r' \ll r$. Now, r is a constant in the integration, so we can take it outside and write the prescription for the potential at A as follows:

$$\begin{aligned}\phi_A = \frac{1}{4\pi\epsilon_0}\Bigg[\frac{1}{r}\underbrace{\int \rho\, dv'}_{K_0} + \frac{1}{r^2}\underbrace{\int r'\cos\theta\, \rho\, dv'}_{K_1} \\ + \frac{1}{r^3}\underbrace{\int r'^2 \frac{(3\cos^2\theta - 1)}{2}\rho\, dv'}_{K_2} + \cdots\Bigg].\end{aligned} \tag{10.9}$$

Each of the integrals above, K_0, K_1, K_2, and so on, has a value that depends only on the structure of the charge distribution, not on the distance to point A. Hence the potential for all points along the z axis can be written as a power series in $1/r$ with constant coefficients:

$$\phi_A = \frac{1}{4\pi\epsilon_0}\left[\frac{K_0}{r} + \frac{K_1}{r^2} + \frac{K_2}{r^3} + \cdots\right]. \tag{10.10}$$

This power series is called the *multipole expansion* of the potential, although we have calculated it only for a point on the z axis here. To finish the problem we would have to get the potential ϕ at all other points, in order to calculate the electric field as $-\text{grad}\,\phi$. We have gone far enough, though, to bring out the essential point: *The behavior of the potential at large distances from the source will be dominated by the first term in the above series whose coefficient is not zero.*

Let us look at these coefficients more closely. The coefficient K_0 is $\int \rho\, dv'$, which is simply the total charge in the distribution. If we have equal amounts of positive and negative charge, as in a neutral molecule,

K_0 will be zero. For a singly ionized molecule, K_0 will have the value e. If K_0 is not zero, then no matter how large K_1, K_2, etc., may be, if we go out to a sufficiently large distance, the term K_0/r will win out. Beyond that, the potential will approach that of a point charge at the origin and so will the field. This is hardly surprising.

Suppose we have a neutral molecule, so that K_0 is equal to zero. Our interest now shifts to the second term, with coefficient $K_1 = \int r' \cos\theta\, \rho\, dv'$. Since $r' \cos\theta$ is simply z', this term measures the relative displacement, in the direction toward A, of the positive and negative charge. It has a nonzero value for the distributions sketched in Fig. 10.4, where the densities of positive and negative charge have been indicated separately. In fact, all the distributions shown have approximately the same value of K_1. Furthermore – and this is a crucial point – *if any charge distribution is neutral, the value of K_1 is independent of the position chosen as origin.* That is, if we replace z' by $z' + z'_0$, in effect shifting the origin, the value of the integral is not changed: $\int (z' + z'_0)\rho\, dv' = \int z' \rho\, dv' + z'_0 \int \rho\, dv'$, and the latter integral is always zero for a neutral distribution.

Evidently, if $K_0 = 0$ and $K_1 \neq 0$, the potential along the z axis will vary asymptotically (that is, with ever-closer approximation as we go out to larger distances) as $1/r^2$. We recognize this dependence on r from the dipole discussion in Section 2.7. We expect the electric field strength to behave asymptotically like $1/r^3$, in contrast with the $1/r^2$ dependence of the field from a point charge. Of course, we have discussed only the potential on the z axis. We will return to the question of the exact form of the field after getting a general view of the situation.

If K_0 and K_1 are both zero, and K_2 is not, the potential will behave like $1/r^3$ at large distances, and the field strength will fall off with the inverse fourth power of the distance. Figure 10.5 shows a charge distribution for which K_0 and K_1 are both zero (and would be zero no matter what direction we had chosen for the z axis), while K_2 is not zero.

The quantities K_0, K_1, K_2,... are related to what are called the *moments* of the charge distribution. Using this language, we call K_0, which is simply the net charge, the *monopole moment,* or *monopole strength.* K_1 is one component of the *dipole moment* of the distribution. The dipole moment has the dimensions (charge) $\times$ (displacement); it is a *vector,* and our K_1 is its z component. The third constant K_2 is related to the *quadrupole moment* of the distribution, the next to the *octupole moment,* and so on. The quadrupole moment is not a vector, but a tensor. The charge distribution shown in Fig. 10.5 has a nonzero quadrupole moment. You can quickly show that $K_2 = 3ea^2$, where a is the distance from each charge to the origin.

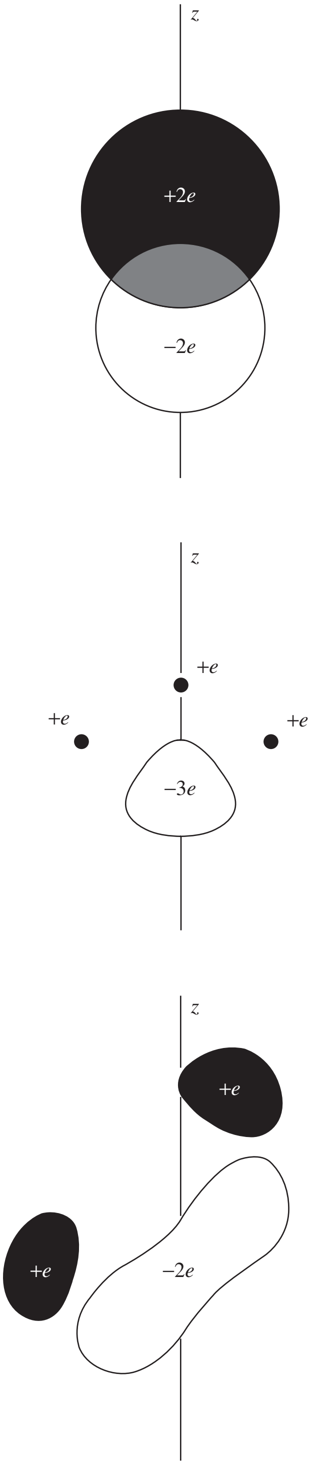


Figure 10.4.
Some charge distributions with $K_0 = 0$, $K_1 \neq 0$. That is, each has net charge zero, but nonzero dipole moment.

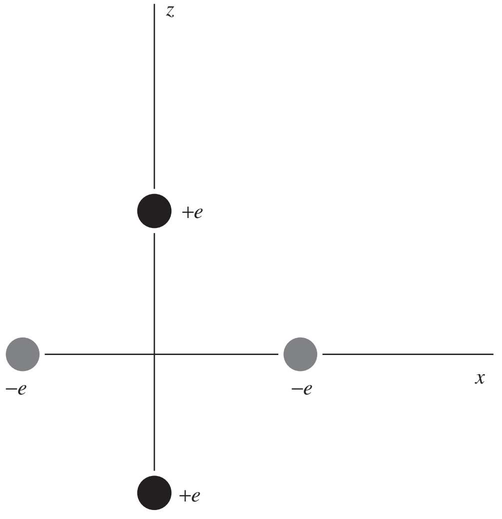


Figure 10.5.
For this distribution of charge, $K_0 = K_1 = 0$, but $K_2 \neq 0$. It is a distribution with nonzero quadrupole moment.

Example (Sphere monopole) The external potential due to a spherical shell with uniform surface charge density is $Q/4\pi\epsilon_0 r$. Therefore, its only nonzero moment is the monopole moment. That is, all of the K_i terms except K_0 in Eq. (10.10) are zero. Using the integral forms given in Eq. (10.9), verify that K_1 and K_2 are zero.

Solution For a surface charge density, the $\rho\, dv'$ in the K_i integrals turns into $\sigma\, da' = \sigma(2\pi R \sin\theta)(R\, d\theta)$. Since we're trying to show that the integrals are zero, the various constants in $\sigma\, da'$ don't matter. Only the angular dependence, $\sin\theta\, d\theta$, is relevant. So we have

$$K_1 \propto \int_0^\pi \cos\theta \sin\theta\, d\theta = -\frac{1}{2}\cos^2\theta \bigg|_0^\pi = 0,$$
$$K_2 \propto \int_0^\pi (3\cos^2\theta - 1)\sin\theta\, d\theta = \big(-\cos^3\theta + \cos\theta\big)\bigg|_0^\pi = 0, \qquad (10.11)$$

as desired. Intuitively, it is clear from symmetry that K_1 is zero; for every bit of charge with height z', there is a corresponding bit of charge with height $-z'$. But it isn't as intuitively obvious that K_2 vanishes.

As mentioned above, K_1 and K_2 are only components of the complete dipole vector and quadrupole tensor. But the other components can likewise be shown to equal zero, as we know they must. If you want to calculate the general form of the complete quadrupole tensor, one way is to write the R in Eq. (10.5) as $R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$, and then perform a Taylor expansion as we did above. See Problem 10.6.

The advantage to us of describing a charge distribution by this hierarchy of moments is that it singles out just those features of the charge distribution that determine the field at a great distance. If we were concerned only with the field in the immediate neighborhood of the distribution, it would be a fruitless exercise. For our main task, understanding what goes on in a dielectric, it turns out that *only* the monopole strength (the net charge) and the dipole strength of the molecular building blocks are important. We can ignore all other moments. And if the building blocks are neutral, we have only their dipole moments to consider.

10.3 The potential and field of a dipole

The dipole contribution to the potential at the point A, at distance r from the origin, is given by $(1/4\pi\epsilon_0 r^2)\int r'\cos\theta\, \rho\, dv'$. We can write $r'\cos\theta$, which is just the projection of $\mathbf{r}'$ on the direction toward A, as $\hat{\mathbf{r}}\cdot\mathbf{r}'$. Thus we can write the potential without reference to any arbitrary axis as

$$\phi_A = \frac{1}{4\pi\epsilon_0 r^2}\int \hat{\mathbf{r}}\cdot\mathbf{r}'\rho\, dv' = \frac{\hat{\mathbf{r}}}{4\pi\epsilon_0 r^2}\cdot\int \mathbf{r}'\rho\, dv', \qquad (10.12)$$

which will serve to give the potential at any point with location $r\hat{\mathbf{r}}$. The integral on the right in Eq. (10.12) is the *dipole moment* of the charge

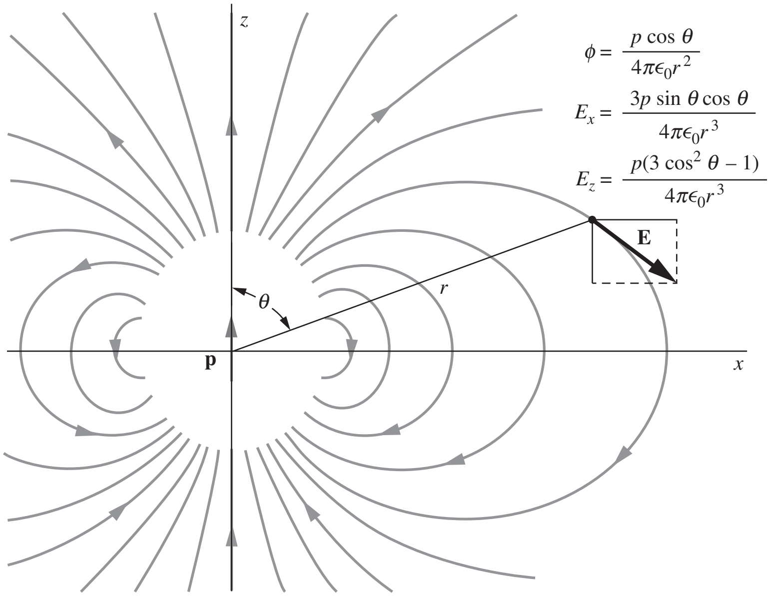


Figure 10.6.
The electric field of a dipole, indicated by some field lines.

distribution. It is a vector, obviously, with the dimensions (charge) × (distance). We shall denote the dipole moment vector by **p**:

$$\boxed{\mathbf{p} = \int \mathbf{r}'\rho\, dv'} \tag{10.13}$$

The dipole moment $p = q\ell$ in Section 2.7 is a special case of this result. If we have two point charges $\pm q$ located at positions $z = \pm\ell/2$, then ρ is nonzero only at these two points. So the integral in Eq. (10.13) becomes a discrete sum: $\mathbf{p} = q(\hat{\mathbf{z}}\ell/2) + (-q)(-\hat{\mathbf{z}}\ell/2) = (q\ell)\hat{\mathbf{z}}$, which agrees with the $p = q\ell$ result in Eq. (2.35). The dipole vector points in the direction from the negative charge to the positive charge.

Using the dipole moment **p**, we can rewrite Eq. (10.12) as

$$\phi(\mathbf{r}) = \frac{\hat{\mathbf{r}}\cdot\mathbf{p}}{4\pi\epsilon_0 r^2}. \tag{10.14}$$

The electric field is the negative gradient of this potential. To see what the dipole field is like, locate a dipole **p** at the origin, pointing in the z direction (Fig. 10.6). With this arrangement,

$$\boxed{\phi = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}} \tag{10.15}$$

in agreement with the result in Eq. (2.35).[1] The potential and the field are, of course, symmetrical around the z axis. Let's work with Cartesian coordinates in the xz plane, where $\cos\theta = z/(x^2+z^2)^{1/2}$. In that plane,

$$\phi = \frac{pz}{4\pi\epsilon_0(x^2+z^2)^{3/2}}. \tag{10.16}$$

The components of the electric field are readily derived:

$$E_x = -\frac{\partial\phi}{\partial x} = \frac{3pxz}{4\pi\epsilon_0(x^2+z^2)^{5/2}} = \frac{3p\sin\theta\cos\theta}{4\pi\epsilon_0 r^3}, \tag{10.17}$$

$$E_z = -\frac{\partial\phi}{\partial z} = \frac{p}{4\pi\epsilon_0}\left[\frac{3z^2}{(x^2+z^2)^{5/2}} - \frac{1}{(x^2+z^2)^{3/2}}\right] = \frac{p(3\cos^2\theta - 1)}{4\pi\epsilon_0 r^3}.$$

The dipole field can be described more simply in the polar coordinates r and θ. Let E_r be the component of $\mathbf{E}$ in the direction of $\hat{\mathbf{r}}$, and let E_θ be the component perpendicular to $\hat{\mathbf{r}}$ in the direction of increasing θ. You can show in Problem 10.4 that Eq. (10.17) implies

$$E_r = \frac{p}{2\pi\epsilon_0 r^3}\cos\theta, \qquad E_\theta = \frac{p}{4\pi\epsilon_0 r^3}\sin\theta, \tag{10.18}$$

in agreement with the result in Eq. (2.36). Alternatively, you can quickly derive Eq. (10.18) directly by working in polar coordinates and taking the negative gradient of the potential given by Eq. (10.15). This is the route we took in Section 2.7.

Proceeding out in any direction from the dipole, we find the electric field strength falling off as $1/r^3$, as we had anticipated. Along the z axis the field is parallel to the dipole moment $\mathbf{p}$, with magnitude $p/2\pi\epsilon_0 r^3$; that is, it has the value $\mathbf{p}/2\pi\epsilon_0 r^3$. In the equatorial plane the field points antiparallel to $\mathbf{p}$ and has the value $-\mathbf{p}/4\pi\epsilon_0 r^3$. This field may remind you of the field in the setup with a point charge over a conducting plane, with its image charge, from Section 3.4. That of course is just the two-charge dipole we discussed in Section 2.7. In Fig. 10.7 we show the field of this pair of charges, mainly to emphasize that the field near the charges is *not* a dipole field. This charge distribution has many multipole moments, indeed infinitely many, so it is only the far field at distances $r \gg s$ that can be represented as a dipole field.

To generate a complete dipole field right into the origin we would have to let s shrink to zero while increasing q without limit so as to keep $p = qs$ finite. This highly singular abstraction is not very interesting. We know that our molecular charge distribution will have complicated near fields, so we could not easily represent the near region in any case. Fortunately we shall not need to.

[1] Note that the angle θ here has a different meaning from the angle θ in Fig. 10.3 and Eqs. (10.5)–(10.9), where it indicated the position of a point in the charge distribution. The present θ indicates the position of a given point (at which we want to calculate ϕ and $\mathbf{E}$) with respect to the dipole direction.

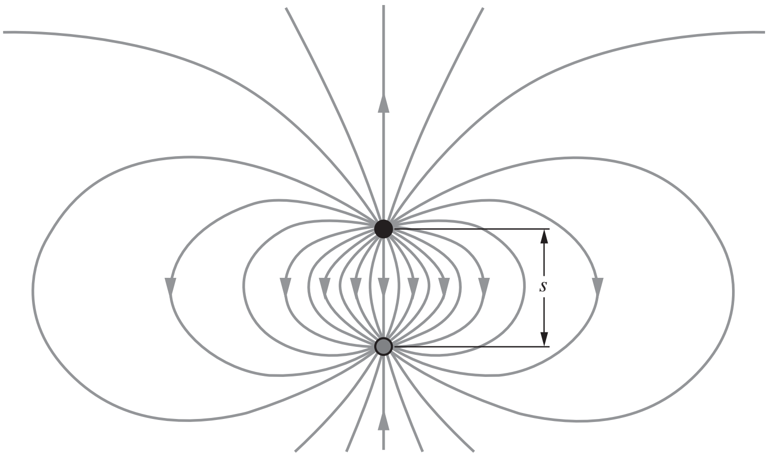


Figure 10.7.
The electric field of a pair of equal and opposite point charges approximates the field of a dipole for distances large compared with the separation s.

10.4 The torque and the force on a dipole in an external field

Suppose two charges, q and $-q$, are mechanically connected so that s, the distance between them, is fixed. You may think of the charges as stuck on the end of a short nonconducting rod of length s. We shall call this object a dipole. Its dipole moment p is simply qs. Let us put the dipole in an external electric field, that is, the field from some other source. The field of the dipole itself does not concern us now. Consider first a uniform electric field, as in Fig. 10.8(a). The positive end of the dipole is pulled toward the right, the negative end toward the left, by a force of strength qE. The net force on the object is zero, and so is the torque, in this position.

A dipole that makes some angle θ with the field direction, as in Fig. 10.8(b), obviously experiences a torque. In general, the torque $\mathbf{N}$ around an axis through some chosen origin is $\mathbf{r} \times \mathbf{F}$, where $\mathbf{F}$ is the force applied at a position $\mathbf{r}$ relative to the origin. Taking the origin in the center of the dipole, so that $r = s/2$, we have

$$\mathbf{N} = \mathbf{r} \times \mathbf{F}_+ + (-\mathbf{r}) \times \mathbf{F}_-. \tag{10.19}$$

$\mathbf{N}$ is a vector perpendicular to the figure, and its magnitude is given by

$$N = \frac{s}{2} qE \sin\theta + \frac{s}{2} qE \sin\theta = sqE \sin\theta = pE \sin\theta. \tag{10.20}$$

This can be written simply as

$$\boxed{\mathbf{N} = \mathbf{p} \times \mathbf{E}} \tag{10.21}$$

When the total force on the dipole is zero, as it is in this case, the torque is independent of the choice of origin (as you should verify), which therefore need not be specified.

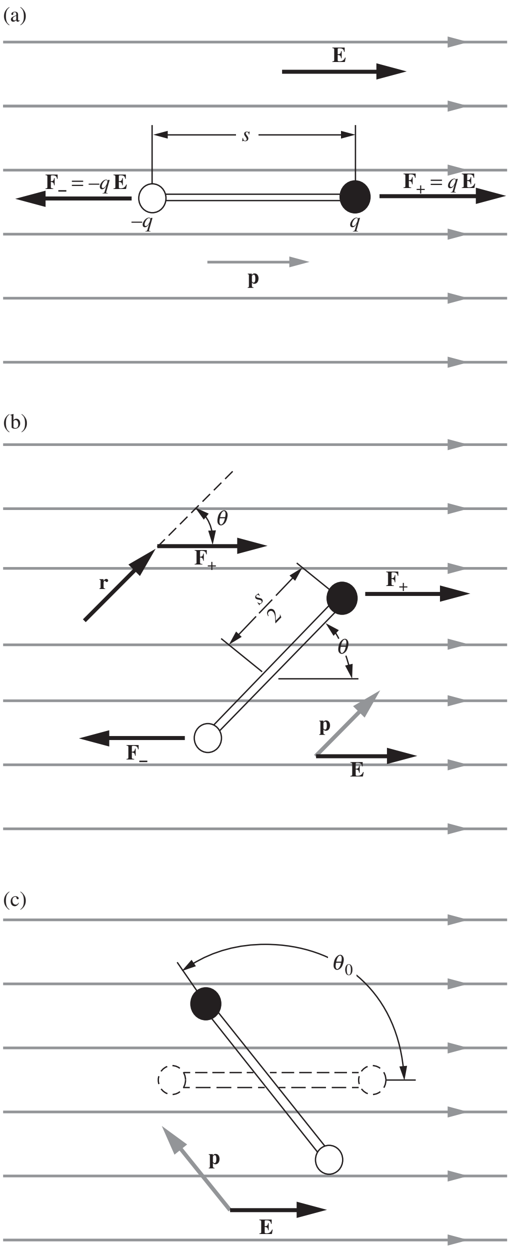


Figure 10.8.
(a) A dipole in a uniform field. (b) The torque on the dipole is $\mathbf{N} = \mathbf{p} \times \mathbf{E}$; the vector $\mathbf{N}$ points into the page. (c) The work done in turning the dipole from an orientation parallel to the field to the orientation shown is $pE(1 - \cos\theta_0)$.

The orientation of the dipole in Fig. 10.8(a) has the lowest energy. Work has to be done to rotate it into any other position. Let us calculate the work required to rotate the dipole from a position parallel to the field, through some angle θ_0, as shown in Fig. 10.8(c). Rotation through an infinitesimal angle $d\theta$ requires an amount of work $N\,d\theta$. Thus the total work done is

$$\int_0^{\theta_0} N\,d\theta = \int_0^{\theta_0} pE\sin\theta\,d\theta = pE(1 - \cos\theta_0). \tag{10.22}$$

This makes sense, because each charge moves a distance $(s/2)(1 - \cos\theta_0)$ against the field. The force is qE, so the work done on each charge is $(qE)(s/2)(1 - \cos\theta_0)$. Doubling this gives the result in Eq. (10.22). To reverse the dipole, turning it end over end, corresponds to $\theta_0 = \pi$ and requires an amount of work equal to $2pE$.

The net force on the dipole in any *uniform* field is zero, obviously, regardless of its orientation. In a nonuniform field the forces on the two ends of the dipole will generally not be exactly equal and opposite, and there will be a net force on the object. A simple example is a dipole in the field of a point charge Q. If the dipole is oriented radially, as in Fig. 10.9(a), with the positive end nearer the positive charge Q, the net force will be outward, and its magnitude will be

$$F = (q)\frac{Q}{4\pi\epsilon_0 r^2} + (-q)\frac{Q}{4\pi\epsilon_0 (r+s)^2}. \tag{10.23}$$

For $s \ll r$, we need only evaluate this to first order in s/r:

$$\begin{aligned} F &= \frac{qQ}{4\pi\epsilon_0 r^2}\left[1 - \frac{1}{\left(1 + \frac{s}{r}\right)^2}\right] \approx \frac{qQ}{4\pi\epsilon_0 r^2}\left[1 - \frac{1}{1 + \frac{2s}{r}}\right] \\ &\approx \frac{qQ}{4\pi\epsilon_0 r^2}\left[1 - \left(1 - \frac{2s}{r}\right)\right] = \frac{sqQ}{2\pi\epsilon_0 r^3}. \end{aligned} \tag{10.24}$$

In terms of the dipole moment p, this is simply

$$F = \frac{pQ}{2\pi\epsilon_0 r^3}. \tag{10.25}$$

With the dipole at right angles to the field, as in Fig. 10.9(b), there is also a force. Now the forces on the two ends, though equal in magnitude, are not exactly opposite in direction. In this case there is a net upward force.

It is not hard to work out a general formula for the force on a dipole in a nonuniform electric field. The force depends essentially on the *gradients* of the various components of the field. In general, the x component of the force on a dipole of moment $\mathbf{p}$ is

$$\boxed{F_x = \mathbf{p} \cdot \text{grad}\, E_x} \tag{10.26}$$

with corresponding formulas for F_y and F_z; see Problem 10.7. All three components can be collected into the concise statement, $\mathbf{F} = (\mathbf{p} \cdot \nabla)\mathbf{E}$.

3.4 ■ MULTIPOLE EXPANSION

3.4.1 ■ Approximate Potentials at Large Distances

If you are very far away from a localized charge distribution, it "looks" like a point charge, and the potential is—to good approximation—$(1/4\pi\epsilon_0)Q/r$, where Q is the total charge. We have often used this as a check on formulas for V. But what if Q is *zero?* You might reply that the potential is then approximately zero, and of course, you're right, in a sense (indeed, the potential at large r is *pretty small* even if Q is *not* zero). But we're looking for something a bit more informative than that.

Example 3.10. A (physical) **electric dipole** consists of two equal and opposite charges ($\pm q$) separated by a distance d. Find the approximate potential at points far from the dipole.

Solution

Let $\mathscr{r}_-$ be the distance from $-q$ and $\mathscr{r}_+$ the distance from $+q$ (Fig. 3.26). Then

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}_+} - \frac{q}{\mathscr{r}_-}\right),$$

and (from the law of cosines)

$$\mathscr{r}_\pm^2 = r^2 + (d/2)^2 \mp rd\cos\theta = r^2\left(1 \mp \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right).$$

We're interested in the régime $r \gg d$, so the third term is negligible, and the binomial expansion yields

$$\frac{1}{\mathscr{r}_\pm} \cong \frac{1}{r}\left(1 \mp \frac{d}{r}\cos\theta\right)^{-1/2} \cong \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right).$$

Thus

$$\frac{1}{\mathscr{r}_+} - \frac{1}{\mathscr{r}_-} \cong \frac{d}{r^2}\cos\theta,$$

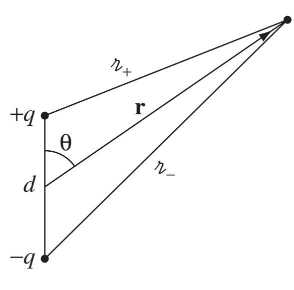


FIGURE 3.26

and hence

$$V(\mathbf{r}) \cong \frac{1}{4\pi\epsilon_0}\frac{qd\cos\theta}{r^2}. \tag{3.90}$$

The potential of a dipole goes like $1/r^2$ at large r; as we might have anticipated, it falls off more rapidly than the potential of a point charge. If we put together a pair of equal and opposite *dipoles* to make a **quadrupole**, the potential goes like $1/r^3$; for back-to-back *quadrupoles* (an **octopole**), it goes like $1/r^4$; and so on. Figure 3.27 summarizes this hierarchy; for completeness I have included the electric **monopole** (point charge), whose potential, of course, goes like $1/r$.

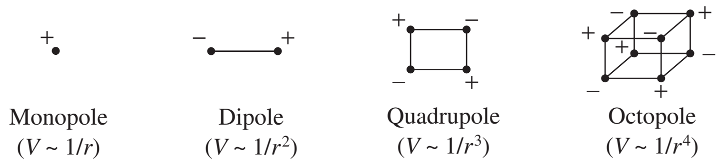


FIGURE 3.27

Example 3.10 pertains to a very special charge configuration. I propose now to develop a systematic expansion for the potential of *any* localized charge distribution, in powers of $1/r$. Figure 3.28 defines the relevant variables; the potential at $\mathbf{r}$ is given by

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}}\rho(\mathbf{r}')\,d\tau'. \tag{3.91}$$

Using the law of cosines,

$$\mathscr{r}^2 = r^2 + (r')^2 - 2rr'\cos\alpha = r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\alpha\right],$$

where α is the angle between $\mathbf{r}$ and $\mathbf{r}'$. Thus

$$\mathscr{r} = r\sqrt{1+\epsilon}, \tag{3.92}$$

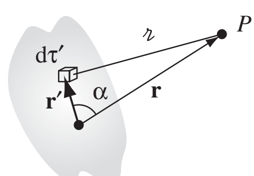


FIGURE 3.28

with

$$\epsilon \equiv \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right).$$

For points well outside the charge distribution, ϵ is much less than 1, and this invites a binomial expansion:

$$\frac{1}{\gamma} = \frac{1}{r}(1+\epsilon)^{-1/2} = \frac{1}{r}\left(1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \ldots\right), \tag{3.93}$$

or, in terms of r, r', and α:

$$\begin{aligned}\frac{1}{\gamma} &= \frac{1}{r}\left[1 - \frac{1}{2}\left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right) + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r} - 2\cos\alpha\right)^2\right.\\ &\qquad \left. - \frac{5}{16}\left(\frac{r'}{r}\right)^3\left(\frac{r'}{r} - 2\cos\alpha\right)^3 + \ldots\right]\\ &= \frac{1}{r}\left[1 + \left(\frac{r'}{r}\right)(\cos\alpha) + \left(\frac{r'}{r}\right)^2\left(\frac{3\cos^2\alpha - 1}{2}\right)\right.\\ &\qquad \left. + \left(\frac{r'}{r}\right)^3\left(\frac{5\cos^3\alpha - 3\cos\alpha}{2}\right) + \ldots\right].\end{aligned}$$

In the last step, I have collected together like powers of (r'/r); surprisingly, their coefficients (the terms in parentheses) are Legendre polynomials! The remarkable result[16] is that

$$\frac{1}{\gamma} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha). \tag{3.94}$$

Substituting this back into Eq. 3.91, and noting that r is a constant, as far as the integration is concerned, I conclude that

$$\boxed{V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{(n+1)}}\int (r')^n P_n(\cos\alpha)\rho(\mathbf{r}')\,d\tau',} \tag{3.95}$$

or, more explicitly,

$$\begin{aligned}V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}&\left[\frac{1}{r}\int \rho(\mathbf{r}')\,d\tau' + \frac{1}{r^2}\int r'\cos\alpha\,\rho(\mathbf{r}')\,d\tau'\right.\\ &\left. + \frac{1}{r^3}\int (r')^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)\rho(\mathbf{r}')\,d\tau' + \ldots\right].\end{aligned} \tag{3.96}$$

[16]This suggests a second way of defining the Legendre polynomials (the first being Rodrigues' formula); $1/\gamma$ is called the **generating function** for Legendre polynomials.

This is the desired result—the **multipole expansion** of V in powers of $1/r$. The first term ($n = 0$) is the monopole contribution (it goes like $1/r$); the second ($n = 1$) is the dipole (it goes like $1/r^2$); the third is quadrupole; the fourth octopole; and so on. Remember that α is the angle between $\mathbf{r}$ and $\mathbf{r}'$, so the integrals depend on the direction to the field point. If you are interested in the potential along the z' axis (or—putting it the other way around—if you orient your $\mathbf{r}'$ coordinates so the z' axis lies along $\mathbf{r}$), then α is the usual polar angle θ'.

As it stands, Eq. 3.95 is *exact,* but it is *useful* primarily as an approximation scheme: the lowest nonzero term in the expansion provides the approximate potential at large r, and the successive terms tell us how to improve the approximation if greater precision is required.

Problem 3.27 A sphere of radius R, centered at the origin, carries charge density

$$\rho(r, \theta) = k\frac{R}{r^2}(R - 2r)\sin\theta,$$

where k is a constant, and r, θ are the usual spherical coordinates. Find the approximate potential for points on the z axis, far from the sphere.

Problem 3.28 A circular ring in the xy plane (radius R, centered at the origin) carries a uniform line charge λ. Find the first three terms ($n = 0, 1, 2$) in the multipole expansion for $V(r, \theta)$.

3.4.2 ■ The Monopole and Dipole Terms

Ordinarily, the multipole expansion is dominated (at large r) by the monopole term:

$$V_{\text{mon}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}, \tag{3.97}$$

where $Q = \int \rho\, d\tau$ is the total charge of the configuration. This is just what we expect for the approximate potential at large distances from the charge. For a *point* charge *at the origin,* V_{mon} is the *exact* potential, not merely a first approximation at large r; in this case, all the higher multipoles vanish.

If the total charge is zero, the dominant term in the potential will be the dipole (unless, of course, it *also* vanishes):

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r'\cos\alpha\,\rho(\mathbf{r}')\,d\tau'.$$

Since α is the angle between $\mathbf{r}'$ and $\mathbf{r}$ (Fig. 3.28),

$$r'\cos\alpha = \hat{\mathbf{r}}\cdot\mathbf{r}',$$

and the dipole potential can be written more succinctly:

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\mathbf{r}}\cdot\int \mathbf{r}'\rho(\mathbf{r}')\,d\tau'.$$

This integral (which does not depend on **r**) is called the **dipole moment** of the distribution:

$$\boxed{\mathbf{p} \equiv \int \mathbf{r}' \rho(\mathbf{r}')\, d\tau',} \tag{3.98}$$

and the dipole contribution to the potential simplifies to

$$\boxed{V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2}.} \tag{3.99}$$

The dipole moment is determined by the geometry (size, shape, and density) of the charge distribution. Equation 3.98 translates in the usual way (Sect. 2.1.4) for point, line, and surface charges. Thus, the dipole moment of a collection of *point* charges is

$$\mathbf{p} = \sum_{i=1}^{n} q_i \mathbf{r}'_i. \tag{3.100}$$

For a **physical dipole** (equal and opposite charges, $\pm q$),

$$\mathbf{p} = q\mathbf{r}'_+ - q\mathbf{r}'_- = q(\mathbf{r}'_+ - \mathbf{r}'_-) = q\mathbf{d}, \tag{3.101}$$

where **d** is the vector from the negative charge to the positive one (Fig. 3.29).

Is this consistent with what we got in Ex. 3.10? Yes: If you put Eq. 3.101 into Eq. 3.99, you recover Eq. 3.90. Notice, however, that this is only the *approximate* potential of the physical dipole—evidently there are higher multipole contributions. Of course, as you go farther and farther away, V_{dip} becomes a better and better approximation, since the higher terms die off more rapidly with increasing r. By the same token, at a fixed r the dipole approximation improves as you shrink the separation d. To construct a **perfect** (point) **dipole** whose potential is given *exactly* by Eq. 3.99, you'd have to let d approach zero. Unfortunately, you then lose the dipole term *too,* unless you simultaneously arrange for q to go to infinity! A *physical* dipole becomes a *pure* dipole, then, in the rather artificial limit $d \to 0$, $q \to \infty$, with the product $qd = p$ held fixed. When someone uses the word "dipole," you can't always tell whether they mean a *physical* dipole (with

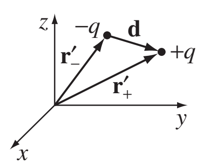


FIGURE 3.29

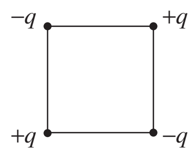


FIGURE 3.30

finite separation between the charges) or an *ideal* (point) dipole. If in doubt, assume that d is small enough (compared to r) that you can safely apply Eq. 3.99.

Dipole moments are *vectors,* and they add accordingly: if you have two dipoles, $\mathbf{p}_1$ and $\mathbf{p}_2$, the total dipole moment is $\mathbf{p}_1 + \mathbf{p}_2$. For instance, with four charges at the corners of a square, as shown in Fig. 3.30, the net dipole moment is zero. You can see this by combining the charges in pairs (vertically, $\downarrow + \uparrow = 0$, or horizontally, $\rightarrow + \leftarrow = 0$) or by adding up the four contributions individually, using Eq. 3.100. This is a *quadrupole,* as I indicated earlier, and its potential is dominated by the quadrupole term in the multipole expansion.

Problem 3.29 Four particles (one of charge q, one of charge $3q$, and two of charge $-2q$) are placed as shown in Fig. 3.31, each a distance a from the origin. Find a simple approximate formula for the potential, valid at points far from the origin. (Express your answer in spherical coordinates.)

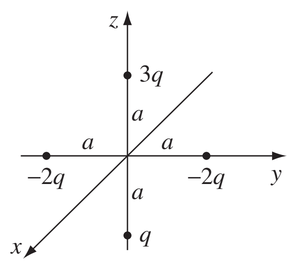


FIGURE 3.31

Problem 3.30 In Ex. 3.9, we derived the exact potential for a spherical shell of radius R, which carries a surface charge $\sigma = k \cos\theta$.

(a) Calculate the dipole moment of this charge distribution.

(b) Find the approximate potential, at points far from the sphere, and compare the exact answer (Eq. 3.87). What can you conclude about the higher multipoles?

Problem 3.31 For the dipole in Ex. 3.10, expand $1/\mathscr{r}_\pm$ to order $(d/r)^3$, and use this to determine the quadrupole and octopole terms in the potential.

3.4.3 ■ Origin of Coordinates in Multipole Expansions

I mentioned earlier that a point charge at the origin constitutes a "pure" monopole. If it is *not* at the origin, it's no longer a pure monopole. For instance, the charge in Fig. 3.32 has a dipole moment $\mathbf{p} = qd\hat{\mathbf{y}}$, and a corresponding dipole term in its potential. The monopole potential $(1/4\pi\epsilon_0)q/r$ is not quite correct for this configuration; rather, the exact potential is $(1/4\pi\epsilon_0)q/\mathscr{r}$. The multipole expansion is, remember, a series in inverse powers of r (the distance to the *origin*), and when we expand $1/\mathscr{r}$, we get *all* powers, not just the first.

So moving the origin (or, what amounts to the same thing, moving the *charge*) can radically alter a multipole expansion. The **monopole moment** Q does not change, since the total charge is obviously independent of the coordinate system. (In Fig. 3.32, the monopole term was unaffected when we moved q away from the origin—it's just that it was no longer the whole story: a dipole term—and for that matter all higher poles—appeared as well.) Ordinarily, the dipole moment *does* change when you shift the origin, but there is an important exception: *If the total charge is zero, then the dipole moment is independent of the choice of origin.* For suppose we displace the origin by an amount $\mathbf{a}$ (Fig. 3.33). The new dipole moment is then

$$\begin{aligned}\bar{\mathbf{p}} &= \int \bar{\mathbf{r}}'\rho(\mathbf{r}')\,d\tau' = \int (\mathbf{r}' - \mathbf{a})\rho(\mathbf{r}')\,d\tau' \\ &= \int \mathbf{r}'\rho(\mathbf{r}')\,d\tau' - \mathbf{a}\int \rho(\mathbf{r}')\,d\tau' = \mathbf{p} - Q\mathbf{a}.\end{aligned}$$

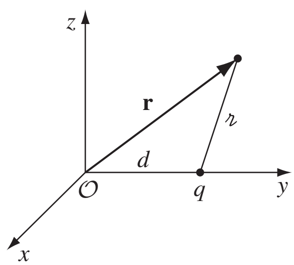


FIGURE 3.32

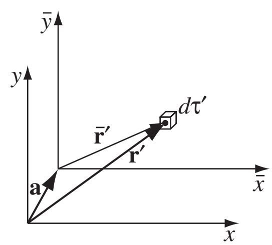


FIGURE 3.33

In particular, if $Q = 0$, then $\bar{\mathbf{p}} = \mathbf{p}$. So if someone asks for the dipole moment in Fig. 3.34(a), you can answer with confidence "$q\mathbf{d}$," but if you're asked for the dipole moment in Fig. 3.34(b), the appropriate response would be "With respect to *what origin*?"

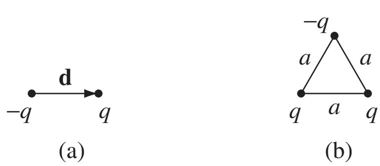
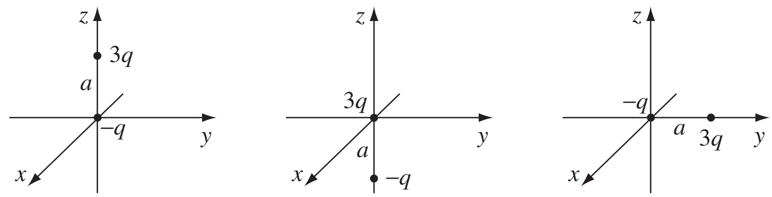


FIGURE 3.34

Problem 3.32 Two point charges, $3q$ and $-q$, are separated by a distance a. For each of the arrangements in Fig. 3.35, find (i) the monopole moment, (ii) the dipole moment, and (iii) the approximate potential (in spherical coordinates) at large r (include both the monopole and dipole contributions).



(a) (b) (c)

FIGURE 3.35

3.4.4 ■ The Electric Field of a Dipole

So far we have worked only with *potentials*. Now I would like to calculate the electric *field* of a (perfect) dipole. If we choose coordinates so that $\mathbf{p}$ is at the origin and points in the z direction (Fig. 3.36), then the potential at r, θ is (Eq. 3.99):

$$V_{\text{dip}}(r, \theta) = \frac{\hat{\mathbf{r}} \cdot \mathbf{p}}{4\pi\epsilon_0 r^2} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}. \tag{3.102}$$

To get the field, we take the negative gradient of V:

$$E_r = -\frac{\partial V}{\partial r} = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3},$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{p\sin\theta}{4\pi\epsilon_0 r^3},$$

$$E_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} = 0.$$

Thus

$$\boxed{\mathbf{E}_{\text{dip}}(r, \theta) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}).} \tag{3.103}$$

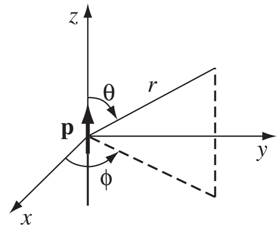
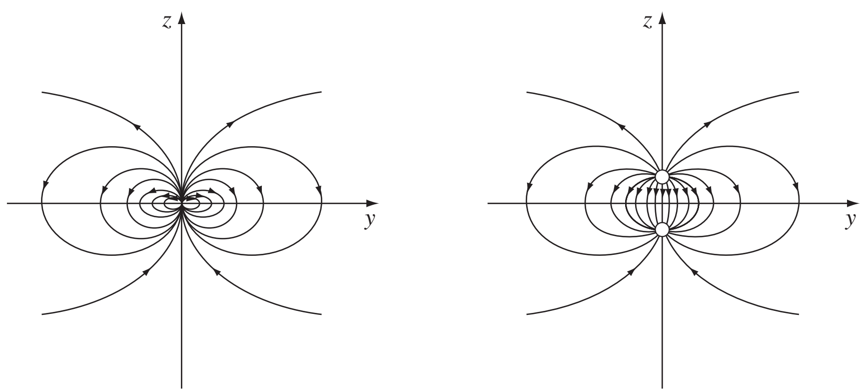


FIGURE 3.36

This formula makes explicit reference to a particular coordinate system (spherical) and assumes a particular orientation for $\mathbf{p}$ (along z). It can be recast in a coordinate-free form, analogous to the potential in Eq. 3.99—see Prob. 3.36.

Notice that the dipole field falls off as the inverse *cube* of r; the *monopole* field $(Q/4\pi\epsilon_0 r^2)\hat{\mathbf{r}}$ goes as the inverse *square,* of course. Quadrupole fields go like $1/r^4$, octopole like $1/r^5$, and so on. (This merely reflects the fact that monopole *potentials* fall off like $1/r$, dipole like $1/r^2$, quadrupole like $1/r^3$, and so on—the gradient introduces another factor of $1/r$.)

Figure 3.37(a) shows the field lines of a "pure" dipole (Eq. 3.103). For comparison, I have also sketched the field lines for a "physical" dipole, in Fig. 3.37(b). Notice how similar the two pictures become if you blot out the central region; up close, however, they are entirely different. Only for points $r \gg d$ does Eq. 3.103 represent a valid approximation to the field of a physical dipole. As I mentioned earlier, this régime can be reached either by going to large r or by squeezing the charges very close together.[17]



(a) Field of a "pure" dipole (b) Field of a "physical" dipole

FIGURE 3.37

[17]Even in the limit, there remains an infinitesimal region at the origin where the field of a physical dipole points in the "wrong" direction, as you can see by "walking" down the z axis in Fig. 3.35(b). If you want to explore this subtle and important point, work Prob. 3.48.

Problem 2.28 Use Eq. 2.29 to calculate the potential inside a uniformly charged solid sphere of radius R and total charge q. Compare your answer to Prob. 2.21.

Problem 2.29 Check that Eq. 2.29 satisfies Poisson's equation, by applying the Laplacian and using Eq. 1.102.

2.3.5 ■ Boundary Conditions

In the typical electrostatic problem you are given a source charge distribution ρ, and you want to find the electric field $\mathbf{E}$ it produces. Unless the symmetry of the problem allows a solution by Gauss's law, it is generally to your advantage to calculate the potential first, as an intermediate step. These are the three fundamental quantities of electrostatics: ρ, $\mathbf{E}$, and V. We have, in the course of our discussion, derived all six formulas interrelating them. These equations are neatly summarized in Fig. 2.35. We began with just two experimental observations: (1) the principle of superposition—a broad general rule applying to *all* electromagnetic forces, and (2) Coulomb's law—the fundamental law of electrostatics. From these, all else followed.

You may have noticed, in studying Exs. 2.5 and 2.6, or working problems such as 2.7, 2.11, and 2.16, that the electric field always undergoes a discontinuity when you cross a surface charge σ. In fact, it is a simple matter to find the *amount* by which $\mathbf{E}$ changes at such a boundary. Suppose we draw a wafer-thin Gaussian pillbox, extending just barely over the edge in each direction (Fig. 2.36). Gauss's law says that

$$\oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}} = \frac{1}{\epsilon_0} \sigma A,$$

where A is the area of the pillbox lid. (If σ varies from point to point or the surface is curved, we must pick A to be extremely small.) Now, the *sides* of the pillbox

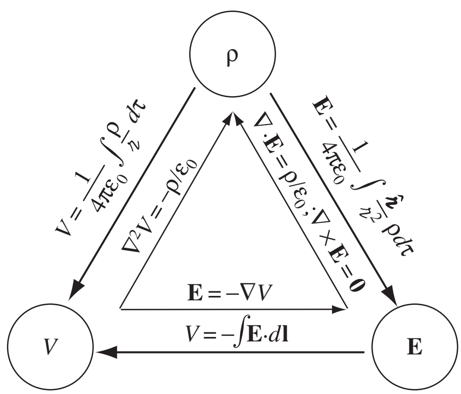


FIGURE 2.35